Journal of Statistical Physics, Vol. 90, Nos. 5/6, 1998

The Triangle Condition for Contact Processes on Homogeneous Trees

Roberto H. Schonmann[1]

Received September 9, 1997; final November 20, 1997

We complete work of C. C. Wu, by showing that for contact processes on homogeneous trees with degree at least 3 the triangle condition is satisfied below the second critical point. In particular it holds at the first critical point and therefore at this critical point the contact process has mean-field critical exponents.

KEY WORDS: Contact process; homogeneous trees; critical behavior; critical exponents; triangle condition.

1. INTRODUCTION AND MAIN RESULTS

The contact process with infection parameter $\lambda > 0$ on a countable graph G of bounded degree is a continuous time Markov process with state space $\{0, 1\}^{\mathscr{V}_G}$, where $\mathscr{V}_G$ is the set of vertices (also called sites) of the graph. Elements of this state space are called configurations. When the configuration at a given site is 1 one says that there is a particle there or that the site is occupied or that the site is infected. Otherwise one says that the site is vacant or healthy. The contact process evolves according to the following local prescription.

(i) A particle at a site gives birth to new ones at each neighboring vacant site at rate λ.

(ii) Particles die at rate 1.

For a more precise description of the model, its construction and background material, we refer the reader to [Lig1] and [Dur].

[1] Department of Mathematics, University of California at Los Angeles, Los Angeles, California 90095; e-mail: rhs@math.ucla.edu.

0022-4715/98/0300-1429$15.00/0

In this paper we will consider the contact process on the homogeneous tree of degree $d+1$, denoted by $\mathbb{T}_d$. The case in which $d=1$ corresponds to the linear chain $\mathbb{Z}$, and will not be considered here, so that we assume that $d \geqslant 2$. A great deal of attention has been given to contact processes on such trees, and a substantial amount of information on its behavior is available from the papers [Pem, MS, MSZ, DS, Wu, Zha, Lig2, Sta, Lig3, LS, SS, and Lal].

We denote by $(\xi_t^\eta)_{t \geqslant 0}$ the contact process started at time 0 from the configuration η. The configuration with a single particle at site x is, in an abuse of notation, denoted by x. An arbitrary site of the tree is called its root, denoted by 0. Configurations are identified with the set of sites where they take the value 1, so that in particular $\varnothing$ is the configuration with no particles. The probability of (global) survival is given by

$$\rho(\lambda) = \mathbb{P}(\xi_t^0 \neq \varnothing, \text{ for all } t \geqslant 0)$$

Similarly, the probability of local survival (sometimes referred to as probability of recurrence) is given by

$$\rho_{\text{local}}(\lambda) = \mathbb{P}(\xi_t^0(0) = 1, \text{ for a unbounded set of values of } t)$$

Interest on the contact process on homogeneous trees with $d \geqslant 2$ stems to a great extent from the fact that there are two distinct critical points. $0 < \lambda_1 < \lambda_2 < \infty$, defined as follows.

$$\lambda_1 = \inf\{\lambda : \rho(\lambda) > 0\}$$

and

$$\lambda_2 = \inf\{\lambda : \rho_{\text{local}}(\lambda) > 0\}$$

In this paper we are concerned with the critical behavior of the process in the vicinity and at the threshold for global survival, λ_1. In [Wu], under the technical assumption that $d \geqslant 5$, this behavior was shown to be of mean-field character, i.e., critical exponents take the branching-process value. We will build on that work and complete it, by removing that technical condition. To state the precise results, we need to introduce a few more definitions.

We suppose that the contact process on $\mathbb{T}_d$, with infection parameter $\lambda > 0$ is constructed in the usual graphical additive fashion, by means of Poisson death marks (at rate 1 for each site) and Poisson arrows (at rate λ for each oriented pair (x, y) such that $\{x, y\}$ is an edge of the graph. We will use $\mathbb{P}_\lambda$ to denote the probability measure corresponding to the graphical construction, and $\mathbb{E}_\lambda$ to denote the corresponding expectation.

Given two space-time points, $(x, s), (y, t) \in \mathscr{V}_{\mathbb{T}_d} \times \mathbb{R}_+$, with $s < t$, we say that there is a path from (x, s) to (y, t) if there is a sequence of times $s = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ and spatial locations $x = x_0, x_1, \ldots, x_n = y$ so that for $i = 1, 2, \ldots, n$ there is an arrow from x_{i-1} to x_i at time t_i and the vertical segments $\{x_i\} \times (t_i, t_{i+1})$ for $i = 0, 1, \ldots, n$ do not contain any death mark. Given a configuration η we set

$$\xi_t^\eta = \{ y \in \mathscr{V}_{\mathbb{T}_d} : \text{there is a path from } (x, 0) \text{ to } (y, t) \text{ for some } x \in \eta \}$$

In this fashion $\{\xi_t^\eta : t \geqslant 0\}$ is a version of the contact process on $\mathbb{T}_d$ started from η. We will use the notation $\{(x, s) \to (y, t)\}$ for the event that there is a path from (x, s) to (y, t) in the graphical construction.

The distance between two sites x and y in $\mathscr{V}_{\mathbb{T}_d}$ will be measured by the minimal length of the paths along neighboring sites which join x to y, and will be denoted by $\|x - y\|$. We will abbreviate $\|x - 0\| = \|x\|$. The distance between two points (x, s) and (y, t) in $\mathscr{V}_{\mathbb{T}_d} \times \mathbb{R}_+$ will be measured by $\|(x, s) - (y, t)\| = \|x - y\| + |t - s|$. The open triangle diagram is now defined by

$$\nabla(\lambda; R) = \sup\{\nabla(\lambda; (z, s)); (z, s) \in \mathscr{V}_{\mathbb{T}_d} \times \mathbb{R}_+, \|(z, s)\| > R\}$$

with

$$\nabla(\lambda; (z, s)) = \sum_{x, y \in \mathscr{V}_{\mathbb{T}_d}} \int_0^\infty dt_1\, \mathbb{P}_\lambda((0, 0) \to (x, t_1)) \times \int_{t_1 \vee s}^\infty dt_2\, \mathbb{P}_\lambda((x, t_1) \to (y, t_2))\, \mathbb{P}_\lambda((z, s) \to (y, t_2))$$

Theorem 1. For the contact process on the homogeneous tree $\mathbb{T}_d$ with $d \geqslant 2$, if $\lambda < \lambda_2$, then

$$\lim_{R \to \infty} \nabla(\lambda; R) = 0$$

Theorem 1 will be proved in Section 2. From the paper [BW] we know that this theorem implies that certain critical exponents take their mean field value, as stated in Corollary 1 below. The susceptibility $\chi(\lambda)$ which appears there is defined as

$$\chi(\lambda) = \mathbb{E}_\lambda \int_0^\infty dt \sum_{x \in \mathscr{V}_{\mathbb{T}_d}} \xi_t^0(x)$$

The notation $f(\lambda) \approx g(\lambda)$ used below means that there exist positive finite constants C_1 and C_2 such that

$$C_1 g(\lambda) \leqslant f(\lambda) \leqslant C_2 g(\lambda)$$

Corollary 1. For the contact process on the homogeneous tree $\mathbb{T}_d$ with $d \geqslant 2$ we have the following mean-field power laws:

$$\rho(\lambda) \approx (\lambda - \lambda_1)^{+1} \quad \text{for} \quad \lambda \geqslant \lambda_1$$

$$\chi(\lambda) \approx (\lambda_1 - \lambda)^{-1} \quad \text{for} \quad \lambda < \lambda_1$$

Our proof of Theorem 1 in Section 2 will give rise to some by-products which will be presented in Section 3.

2. PROOF OF THEOREM 1

Before we can start the proof of Theorem 1, we need to recall some definitions and results from [Lig3, LS and La1]. Denote as site 1 one of the neighbors of the site 0; then denote as site 2 one of the neighbors of site 1 which is different from site 0; successively denote as site $n+1$ one of the neighbors of site n which is different from site $n-1$. Note that site n is at distance n from the origin and that the sites 0, 1,... are the sites of a subgraph of $\mathbb{T}_d$ which is isomorphic to the semi-infinite linear chain $\mathbb{Z}^+$. Set

$$u_n(\lambda) = \mathbb{P}_\lambda(\xi_t^0(n) = 1 \text{ for some } t \geqslant 0)$$

From the strong Markov property, used at the first time that the site m becomes infected, in conjunction with the property of attractiveness of the contact process one obtains the inequality

$$u_{n+m}(\lambda) \geqslant u_n(\lambda)\, u_m(\lambda)$$

From this inequality it follows that the following limit exists.

$$\lim_{n \to \infty} (u_n(\lambda))^{1/n} = \sup\{(u_n(\lambda))^{1/n} : n \geqslant 1\} = \beta(\lambda) \tag{2.1}$$

Note that

$$u_n(\lambda) \leqslant (\beta(\lambda))^n \tag{2.2}$$

for $n \geqslant 0$.

Similarly we have the existence of the limit

$$\lim_{t \to \infty} (\mathbb{P}_\lambda(\xi_t^0(0) = 1))^{1/t} = \sup\{(\mathbb{P}_\lambda(\xi_t^0(0) = 1))^{1/t} : t > 0\} = \eta(\lambda) \quad (2.3)$$

and the inequality

$$\mathbb{P}_\lambda(\xi_t^0(0) = 1) \leqslant (\eta(\lambda))^t \quad (2.4)$$

for $t \geqslant 0$.

From [LS] and [Lal], we know that for $\lambda < \lambda_2$,

$$\beta(\lambda) < 1/\sqrt{d} \qquad \text{and} \qquad \eta(\lambda) < 1 \quad (2.5)$$

We also need to introduce the following object.

$$\bar{u}_n(\lambda) = \int_0^\infty \mathbb{P}_\lambda(\xi_t^0(n) = 1)\, dt = \int_0^\infty \mathbb{P}_\lambda((0, 0) \to (n, t))\, dt$$

Note that

$$\chi(\lambda) = \bar{u}_0(\lambda) + \sum_{n \geqslant 1} (d+1)\, d^{n-1} \bar{u}_n(\lambda) \quad (2.6)$$

Lemma 1. Given $\varepsilon > 0$, there exists $C < \infty$ such that for all $\lambda \in [\varepsilon, \lambda_2 - \varepsilon]$ and $n \geqslant 0$,

$$\bar{u}_n(\lambda) \leqslant C(n+1)\, u_n(\lambda) \leqslant C(n+1)(\beta(\lambda))^n$$

Proof. Let a_ε be the probability for the contact process with infection parameter ε, that between time 0 and 1 there is no death mark at the origin, and there is an arrow from the origin to its neighbor 1. Clearly $a_\varepsilon = e^{-1}(1 - e^{-\varepsilon}) > 0$, and for $\lambda \geqslant \varepsilon$

$$u_n(\lambda) \geqslant (a_\varepsilon)^n \quad (2.7)$$

Clearly also

$$\mathbb{P}_\lambda(\xi_{t+n}^0(0) = 1) \geqslant (a_\varepsilon)^n\, \mathbb{P}_\lambda(\xi_t^0(n) = 1) \quad (2.8)$$

Comparing this with (2.4) yields, in case $\lambda \leqslant \lambda_2 - \varepsilon$,

$$\mathbb{P}_\lambda(\xi_t^0(n) = 1) \leqslant (a_\varepsilon)^{-n}\, (\eta(\lambda_2 - \varepsilon))^{t+n} \leqslant (a_\varepsilon)^{-n}\, (\eta(\lambda_2 - \varepsilon))^t$$

Therefore, for arbitrary $K>0$,

$$\bar{u}_n(\lambda)=\int_0^\infty \mathbb{P}_\lambda(\xi_t^0(n)=1)\,dt \leqslant u_n(\lambda)\,Kn+\int_{Kn}^\infty \mathbb{P}_\lambda(\xi_t^0(n)=1)\,dt$$
$$\leqslant Knu_n(\lambda)+(a_\varepsilon)^{-n}\,(\eta(\lambda_2-\varepsilon))^{Kn}/\log(1/\eta(\lambda_2-\varepsilon))$$

where the fact that $\eta(\lambda_2-\varepsilon)<1$ was used. Using again this fact and referring to (2.7), one concludes that if K is large enough and $C=2K$ we have

$$\bar{u}_n(\lambda)\leqslant 2Knu_n(\lambda)\leqslant Cnu_n(\lambda)\leqslant C(n+1)\,u_n(\lambda)$$

for $n\geqslant 1$. That for large C, we have $\bar{u}_n(\lambda)\leqslant C(n+1)\,u_n(\lambda)$ also in case $n=0$ is trivial, since $\eta(\lambda)\leqslant\eta(\lambda_2-\varepsilon)<1$. The proof is complete by referring to (2.2). ∎

Proof of Theorem 1. We will explain how Lemma 1 can be used to adapt the proof in [Wu] to all $d\geqslant 2$.

Suppose that $\lambda<\lambda_2$. Note that from Lemma 1 and (2.5) it follows that there exists $\beta'\in(\beta(\lambda),1/\sqrt{d})$ and $C_1<\infty$ such that

$$\bar{u}_n(\lambda)\leqslant C_1(\beta')^n \tag{2.9}$$

for all n. This inequality replaces Lemma 2 in [Wu], and confirms the conjecture in that paper stated immediately before that lemma.

Now we rewrite display (40) in [Wu]. We still use his Lemma 3 as he did, but in the last step, we use (2.9) to obtain the following.

$$\begin{aligned}
\nabla(\lambda;(z,s)) &= \sum_{x,y\in\mathscr{V}_{T_d}}\int_0^\infty dt_1\,\mathbb{P}_\lambda((0,0)\to(x,t_1))\\
&\quad\times\int_{t_1\vee s}^\infty dt_2\,\mathbb{P}_\lambda((x,t_1)\to(y,t_2))\,\mathbb{P}_\lambda((z,s)\to(y,t_2))\\
&\leqslant e^4\sum_{x,y\in\mathscr{V}_{T_d}}\int_0^\infty dt_1\,\mathbb{P}_\lambda((0,0)\to(x,t_1))\\
&\quad\times\int_{t_1}^\infty dt_2\,\mathbb{P}_\lambda((x,t_1)\to(y,t_2))\int_s^\infty dt_3\,\mathbb{P}_\lambda((z,s)\to(y,t_3))\\
&= e^4\sum_{x,y\in\mathscr{V}_{T_d}}\bar{u}_{\|x\|}(\lambda)\,\bar{u}_{\|y-x\|}(\lambda)\,\bar{u}_{\|z-y\|}(\lambda)\\
&\leqslant C_2\sum_{x,y\in\mathscr{V}_{T_d}}(\beta')^{\|x\|}\,(\beta')^{\|y-x\|}\,(\beta')^{\|z-y\|}
\end{aligned}$$

To see that the last expression above vanishes as $z \to \infty$, we can refer to Lemma 1 in [Wu]. For this, the reader Should compare the statement of that lemma in [Wu] with his Eq. (23).

At this point we know that

$$\lim_{\|z\| \to \infty} \nabla(\lambda; (z, s)) = 0 \qquad \text{uniformly for} \quad s \in [0, \infty)$$

So it remains to show that for any fixed $z \in \mathbb{T}_d$,

$$\lim_{s \to \infty} \nabla(\lambda; (z, s)) = 0$$

But using the results above, this can now be done exactly in the same way as in [Wu] (see the final part of his proof of his Theorem 1). ∎

3. SOME PROPERTIES OF β

It is clear that $\beta(\lambda) = 1$ for $\lambda > \lambda_2$. A great deal of information about the behavior of $\beta(\lambda)$ as a function of λ in the interval $(0, \lambda_2]$ has been obtained in the papers [Lig3, LS and Lal]. We will review these known results at the end of this section; when we combine them with some new results which will be proved here.

Before we can state these new results, we need to introduce some definitions. We will let the random variable X_n be the total number of arrows which point from an infected site to the site n in the process $(\xi_t^0 : t \geq 0)$. And we also define

$$I_0(\lambda) = \mathbb{E}_\lambda(X_0) + 1$$
$$I_n(\lambda) = \mathbb{E}_\lambda(X_n), \qquad n \geq 1$$

(The extra 1 in the definition of $I_0(\lambda)$ is needed to make certain statements below true; it may intuitively be seen as accounting for the particle initially at the origin.)

Clearly we have

$$u_n(\lambda) \leq I_n(\lambda) \tag{3.1}$$

for $n \geq 0$, and

$$I_n(\lambda) = \lambda(\bar{u}_{n-1}(\lambda) + d\bar{u}_{n+1}(\lambda)) \tag{3.2}$$

for $n \geq 1$.

We claim that also

$$I_{m+n}(\lambda) \leqslant I_m(\lambda)\, I_n(\lambda) \tag{3.3}$$

for $m, n \geqslant 0$. To justify (3.3), we first note that in case $m=0$ or $n=0$ this claim is trivially true. For the other cases we compare the contact process starting with a single particle at the origin with a multitype contact process described as follows. Particles can be of type $0, 1, 2,\ldots$, with the particle initially at the origin being of type 0. Particles of types $1, 2,\ldots$ evolve as independent contact processes, independent also of the behavior of particles of type 0 (in particular particles of different type can coexist at the same site at the same time). Particles of type 0 evolve as a contact process, except for the fact that they cannot infect the site m. The first time when there is an attempt by a particle of type 0 to infect the site m, a particle of type 1 is created at site m; the second time when there is an attempt by a particle of type 0 to infect the site m, a particle of type 2 is created at site m; and so on. This multitype contact process dominates the contact process in the sense that whenever there is a contact process particle at a site, there will be some multitype contact process particle (of some type) also at that site. Let X'_m be the total number of times when a type 0 particle tries to infect the site m in the multitype contact process (this equals the number of other types of particles ever created in this process). Let X''_{m+n} be the the total number of times when a particle of any type tries to infect the site $m+n$ in the multitype contact process (due to the geometry of a tree, we are only talking about particles of types different from 0 here). Now note that

$$\begin{aligned} I_{m+n}(\lambda) &\leqslant \mathbb{E}_\lambda(X''_{m+n}) = \mathbb{E}_\lambda(\mathbb{E}_\lambda(X''_{m+n} \mid X'_m)) = \mathbb{E}_\lambda(X'_m \mathbb{E}_\lambda(X_n)) \\ &= \mathbb{E}_\lambda(X'_m)\, \mathbb{E}_\lambda(X_n) \leqslant \mathbb{E}_\lambda(X_m)\, \mathbb{E}_\lambda(X_n) = I_m(\lambda)\, I_n(\lambda) \end{aligned}$$

The inequality (3.3) yields

$$\lim_{n \to \infty} (I_n(\lambda))^{1/n} = \inf\{(I_n(\lambda))^{1/n} : n \geqslant 1\} = \tilde{\beta}(\lambda) \tag{3.4}$$

Note that

$$I_n(\lambda) \geqslant (\tilde{\beta}(\lambda))^n \tag{3.5}$$

for $n \geqslant 0$.

Theorem 2. The function $\beta(\cdot)$ is continuous on the interval $(0, \lambda_2]$. For $\lambda < \lambda_2$

$$\lim_{n \to \infty} (\bar{u}_n(\lambda))^{1/n} = \lim_{n \to \infty} (I_n(\lambda))^{1/n} = \lim_{n \to \infty} (u_n(\lambda))^{1/n} = \beta(\lambda)$$

Moreover, given $\varepsilon > 0$ there exists $C(\varepsilon) \in (0, \infty)$ such that for all $\lambda \in [\varepsilon, \lambda_2 - \varepsilon]$ we have

$$\frac{(\beta(\lambda))^n}{C(\varepsilon)} \leqslant \bar{u}_n(\lambda) \leqslant C(\varepsilon)(n+1)(\beta(\lambda))^n$$

$$(\beta(\lambda))^n \leqslant I_n(\lambda) \leqslant C(\varepsilon)(n+1)(\beta(\lambda))^n$$

and

$$\frac{(\beta(\lambda))^n}{C(\varepsilon)(n+1)} \leqslant u_n(\lambda) \leqslant (\beta(\lambda))^n$$

Proof. Suppose that $\varepsilon > 0$ is fixed and that $\lambda \in [\varepsilon, \lambda_2 - \varepsilon]$. Using the notation introduced in the beginning of the proof of Lemma 1, we have

$$\begin{aligned} \bar{u}_n(\lambda) &\geqslant \int_1^\infty \mathbb{P}_\lambda(\xi_t^0(n) = 1)\, dt \geqslant \int_1^\infty a_\varepsilon \mathbb{P}_\lambda(\xi_{t-1}^0(n-1) = 1)\, dt \\ &= \int_0^\infty a_\varepsilon \mathbb{P}_\lambda(\xi_t^0(n-1) = 1)\, dt = a_\varepsilon \bar{u}_{n-1}(\lambda) \end{aligned} \tag{3.6}$$

The same argument also gives

$$\bar{u}_n(\lambda) \geqslant a_\varepsilon \bar{u}_{n+1}(\lambda) \tag{3.7}$$

From (3.2), (3.6) and (3.7), we have $I_n(\lambda) \leqslant C'(\varepsilon)\, \bar{u}_n(\lambda)$, and combining this with Lemma 1 and (3.1) yields

$$I_n(\lambda) \leqslant C'(\varepsilon)\, \bar{u}_n(\lambda) \leqslant C''(\varepsilon)(n+1)\, u_n(\lambda) \leqslant C''(\varepsilon)(n+1)\, I_n(\lambda) \tag{3.8}$$

This chain of inequalities and the fact that $\varepsilon > 0$ is arbitrary given us

$$\tilde{\beta}(\lambda) = \beta(\lambda) \tag{3.9}$$

for $\lambda \in (0, \lambda_2)$. Combining it also with (2.2) and (3.5) given us all the claims in the theorem, except for the continuity of $\beta(\cdot)$ in $(0, \lambda_2]$. The proof will therefore be complete once we show that

$$\beta(\cdot) \text{ is left-continuous on } (0, \infty) \tag{3.10}$$

and

$$\beta(\cdot) \text{ is right-continuous on } (0, \lambda_2) \tag{3.11}$$

To show the first of these claims, note that if for $T > 0$ we set

$$u_n^T = u_n^T(\lambda) = \mathbb{P}_\lambda(\xi_t^0(n) = 1 \text{ for some } t \in [0, T])$$

then

$$\beta(\lambda) = \sup_{n \geq 1} (u_n(\lambda))^{1/n} = \sup_{n \geq 1} \sup_{T \geq 0} (u_n^T(\lambda))^{1/n}$$

But $u_n^T(\cdot)$ is clearly a continuous function, and therefore $\beta(\cdot)$ is lower-semi-continuous. Since $\beta(\cdot)$ is clearly non-decreasing, it is also left-continuous.

To show (3.11) define

$$\bar{u}_n^T(\lambda) = \int_0^T \mathbb{P}_\lambda(\xi_t^0(n) = 1)\, dt$$

For fixed λ, $\bar{u}_n^T(\lambda)$ approaches $\bar{u}_n(\lambda)$ as $T \to \infty$, in a fashion which can be controlled uniformly in $\lambda \in [\varepsilon, \lambda_2 - \varepsilon]$ as follows, using (2.8) and (2.4),

$$0 \leq \bar{u}_n(\lambda) - \bar{u}_n^T(\lambda) = \int_T^\infty \mathbb{P}_\lambda(\xi_t^0(n) = 1)\, dt \leq \left(\frac{1}{a_\varepsilon}\right)^n \int_T^\infty \mathbb{P}_\lambda(\xi_{t+n}^0(0) = 1)\, dt$$

$$\leq \left(\frac{1}{a_\varepsilon}\right)^n \int_T^\infty (\eta(\lambda_2 - \varepsilon))^{t+n}\, dt$$

Since $\eta(\lambda_2 - \varepsilon) < 1$, by (2.5), the last expression above vanishes as $T \to \infty$, showing that $\bar{u}_n^T(\cdot) \to \bar{u}_n(\cdot)$, uniformly on $[\varepsilon, \lambda_2 - \varepsilon]$. As a function of λ, $\bar{u}_n^T(\lambda)$ is clearly continuous, hence $\bar{u}_n(\cdot)$ is also continuous on $[\varepsilon, \lambda_2 - \varepsilon]$.

From the part of the theorem which has already been proved, we know that $(C(\varepsilon)\, \bar{u}_n(\lambda))^{1/n} \geq \beta(\lambda)$ and $\lim_{n \to \infty} (C(\varepsilon)\, \bar{u}_n(\lambda))^{1/n} = \beta(\lambda)$. Hence

$$\beta(\lambda) = \inf_{n \geq 1} (C(\varepsilon)\, \bar{u}_n(\lambda))^{1/n}$$

Therefore $\beta(\cdot)$ is upper-semi-continuous on $[\varepsilon, \lambda_2 - \varepsilon]$. Since $\beta(\cdot)$ is non-decreasing, it is also right-continuous on this interval. Since $\varepsilon > 0$ is arbitrary, (3.11) is assured. ∎

Combining results in [Lig3, LS and Lal] with those in the present paper we know now that the function $\beta(\cdot)$, which is identically 1 on (λ_2, ∞) is strictly increasing and continuous on $(0, \lambda_2]$ and satisfies also:

$$\lim_{\lambda \searrow 0} \beta(\lambda) = 0 \tag{3.12}$$

$$\beta(\lambda_1) = 1/d \tag{3.13}$$

and

$$\beta(\lambda_2) = 1/\sqrt{d} \tag{3.14}$$

The strict monotonicity, as well as (3.14) are proved in [Lal]. The claim (3.13) follows, for instance, from combining the following: the continuity of $\beta(\cdot)$ at λ_1, the result on the critical behavior of $\chi(\lambda)$ near λ_1, as given in Corollary 1, Eq. (2.6), and the relation between $\bar{u}(\lambda)$ and $\beta(\lambda)$, as given in Theorem 2. Alternatively, observe that [Lig3] proved that $\beta(\lambda) \leqslant 1/d$ for $\lambda < \lambda_1$ and that for $\lambda > \lambda_1$ clearly $\beta(\lambda) \geqslant 1/d$ (since $\beta(\lambda) < 1/d$ implies $\mathbb{E}_\lambda$ (number of sites ever to be infected) $< \infty$, and this implies $\rho(\lambda) = 0$); the continuity of $\beta(\cdot)$ at λ_1 yields then (3.13). Finally (3.12) is easy to verify using the standard comparison of the contact process with a branching random walk, obtained by allowing multiple occupancy of each site by particles which create offspring and may die independently.

ACKNOWLEDGMENTS

Thanks are given to Tom Liggett and to a referee for their comments which led to improvements to this paper. This work was partially supported by the NSF through grant DMS-9703814.

REFERENCES

[BW] D. J. Barsky and C. C. Wu, Critical exponents for the contact process under the triangle condition, *J. Stat. Phys.* (to appear).

[Dur] R. Durrett, *Lecture Notes on Interacting Particle Systems and Percolation* (Wadsworth & Brooks/Cole, 1988).

[DS] R. Durrett and R. B. Schinazi, Intermediate phase for the contact process on a tree, *Ann. Probab.* **23**:668–673 (1995).

[Lal] S. Lalley, Growth profile and invariant measures for the weakly supercritical contact process on a homogeneous tree, preprint 1997.

[LS] S. Lalley and T. Sellke, Limit set of a weakly supercritical contact process on a homogeneous tree, preprint 1996.

[Lig1] T. M. Liggett, *Interacting Particle Systems* (Springer Verlag, 1985).

[Lig2] T. M. Liggett, Multiple transition points for the contact process on the binary tree, *Ann. Probab.* **24**:1675–1710 (1996).

[Lig3] T. M. Liggett, Branching random walks and contact processes on homogeneous trees, *Probab. Theory and Related Fields* **106**:495–519 (1996).

[MS] N. Madras and R. B. Schinazi, Branching random walk on trees, *Stochastic Processes and Their Applications* **42**:255–267 (1992).

[MSZ] G. J. Morrow, R. B. Schinazi, and Y. Zhang, The critical contact process on a homogeneous tree, *J. Appl. Probab.* **31**:250–255 (1994).

[Pem] R. Pemantle, The contact process on trees, *Ann. Probab.* **20**:2089–2116 (1992).

[SS] M. Salzano and R. H. Schonmann, A new proof that for the contact process on homogeneous trees local survival implies complete convergence, preprint 1997, *Ann. Probab.* (to appear).

[Sta] A. M. Stacey, The existence of an intermediate phase for the contact process on trees, *Ann. Probab.* **24**:1711–1726 (1996).

[Wu] C. C. Wu, The contact process on a tree-behavior near the first transition, *Stochastic Processes and their Applications* **57**:99–112 (1995).

[Zha] Y. Zhang, The complete convergence theorem of the contact process on trees, *Ann. Probab.* **24**:1408–1443 (1996).

Communicated by J. L. Lebowitz